Quotient Spaces

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Overview

Definition 1.

Let W be a subspace of a vector space V over F. By a **coset** *of W in V we mean a set of the form*

$$v + W := \left\{ v + w : w \in W \right\}$$

for some $v \in V$.

We denote the set of cosets of W in V by V/W.

Example 2.

Let $V = \mathbb{R}^2$ and $W = \mathbb{R}e_1 = \left\{ (x, 0) : x \in \mathbb{R} \right\}$. Take any $v = (a, b) \in \mathbb{R}^2$. Then

$$v + W = \left\{ (c, b) : c \in \mathbb{R} \right\}$$

is the line through (a, b) parallel to the x-axis.
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Example 3.

Let $V = \mathbb{R}^3$ *and* $W = \left\{(x, y, 0) : x, y \in \mathbb{R}\right\}$ *be the xy-plane. Then the coset* $v + W$ *for any* $v \in \mathbb{R}^3$ *is the plane parallel to the xy-plane through the point* $v = (a, b, c)$ *at "height"* c.

Exercise 4.

Let $W := \left\{(x, y) \in \mathbb{R}^2 : ax + yb = 0\right\}$ *for a fixed* $(a, b) \in \mathbb{R}^2 \backslash (0, 0)$. *Show that* W *is a one-dimensional subspace of* $V = \mathbb{R}^2$ *and that the cosets of* W *in* V *are the lines parallel to the line* $ax + by = 0$ *and hence are given by* $ax + by = c$ *for* $c \in \mathbb{R}$.

Lemma 5.

Let W be a subspace of a vector space over F. Let $v_i + W$ be cosets for $i = 1, 2$. Then exactly one of the following is true :

(a) $(v_1 + W) \cap (v_2 + W) = \emptyset$

(b) $v_1 + W = v_2 + W$.

Moreover, $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Definition 6.

Let $\xi \in V/W$ be a coset. If $\xi = v + W$, then v is called a **representative** *of ξ.*

If $\xi = v + W$ as well as $\xi = v + W$, then $u - v$ is an element of W.

Any two representatives of a coset of W differ by an element of W.

In particular, $x + W = W$ if and only if $x \in W$.

"Addition" of two cosets $\xi_1 := v_i + W$ in V/W

Let v_i be a representative of ξ_i for $i = 1, 2$. Then we defind $\xi_1 + \xi_2$ to be the coset whose representative is $v_1 + v_2$. That is,

$$\xi_1 + \xi_2 = (v_1 + v_2) + W.$$

We need to show that this coset $\xi_1 + \xi_2$ is defined without any ambiguity. This is usually called "**the well-definedness**" of the concept.

Is "the sum of two cosets" well-defined?

Is the sum depending on representatives choosen on each coset ?

Suppose that there are two representatives v_i and u_i for ξ_1, $i = 1, 2$.

Then the sum $\xi_1 + \xi_2$ is the coset $(v_1 + v_2) + W$ and $(u_1 + u_2) + W$.

“Addition” of two cosets $\xi_1 := v_i + W$ in V/W

Claim :

$$(v_1 + v_2) + W = (u_1 + u_2) + W. \tag{1}$$

The equation (1) holds iff $(v_1 + v_2) + W = (u_1 + u_2) + W$
iff $(v_1 - u_1) + (v_2 - u_2) \in W$.

But

$$(v_1 - u_1) + (v_2 - u_2) \in W$$

is true, since v_i and u_i are representatives of the same coset and hence

$$u_i - v_i \in W.$$

Since W is a subspace $(v_1 - u_1) + (v_2 - u_2) \in W$.

Thus to defind the sumof two cosets we may use any two representatives.

"Scalar Multiplication" on V/W

If $\alpha \in F$ and $\xi = v + W \in V/W$, then $(\alpha\xi) := \alpha v + W$.

Example 7.

Show that the operation "scalar multiplication" is well-defined.

Dimension of Quotient Space

Theorem 8.

Let W be a subspace of V. Let V/W denote the set of cosets of V with respect to W. The following operations are well-defined :

(a) $\xi_1 + \xi_2 = (v_1 + v_2) + W$, *where* $\xi_i := v_i + W \in V/W$.

(b) $\alpha\xi = (\alpha v) + W$, *where* $\xi = v + W \in V/W$.

V/W is called the **quotient space** *of V with respect to W.*

Theorem 9.

Let V be a finite-dimensional vector space and W a subspace of V. Then

$$\dim V/W = \dim V - \dim W.$$

References

- S. Kumaresan, "*Linear Algebra – A Geometric Approach*", Prentice-Hall of India, 2011 (pages mainly from 47 to 52).